

# THE MATZOH BALL SOUP PROBLEM: A COMPLETE CHARACTERIZATION

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**ABSTRACT.** We characterize all the solutions of the heat equation that have their (spatial) equipotential surfaces which do not vary with the time. Such solutions are either isoparametric or split in space-time. The result gives a final answer to a problem raised by M. S. Klamkin, extended by G. Alessandrini, and that was named the *Matzoh Ball Soup Problem* by L. Zalcman. Similar results can also be drawn for a class of quasi-linear parabolic partial differential equations with coefficients which are homogeneous functions of the gradient variable. This class contains the (isotropic or anisotropic) evolution  $p$ -Laplace and normalized  $p$ -Laplace equations.

## 1. INTRODUCTION

In this paper we settle a question raised by M.S. Klamkin in [Kl] and extended in [Al2].

**Klamkin's conjecture (1964).** Consider the heat conduction problem for a solid  $\Omega$ ,

$$u_t = \Delta u \quad \text{in } \Omega \times (0, \infty).$$

Initially,  $u = 0$ . On the boundary  $u = 1$ . The solution to the problem is well-known for a sphere and, as to be expected, it is radially symmetric. Consequently, the equipotential surfaces do not vary with the time (the temperature on them, of course, varies). It is conjectured for the boundary value problem above, that the sphere is the only bounded solid having the property of invariant equipotential surfaces. If we allow unbounded solids, then another solution is the infinite right circular cylinder which corresponds to the spherical solution in two-dimensions.

L. Zalcman [Za] included this problem in a list of questions about the ball and named it the *Matzoh Ball Soup* problem. For the case of a bounded solid, the conjecture was given a positive answer by G. Alessandrini [Al1]: *the ball is the only bounded solid having the property of invariant equipotential surfaces*.

In [MS1], the second author of this paper and S. Sakaguchi showed that to obtain the spherical symmetry of the solid in Klamkin's setting, it is

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enough to require that the solution has *only one* invariant equipotential surface (provided this surface is a  $C^1$ -regular boundary of domain). In a subsequent series of papers, the same authors extended their result in several directions: spherical symmetry also holds for certain evolution nonlinear equations ([MS2, MS4, MS6, MS7]); a hyperplane can be characterized as an invariant equipotential surface in the case of an unbounded solid that satisfies suitable sufficient conditions ([MS3, MS5]); for a certain Cauchy problem, a helicoid is a possible invariant equipotential surface ([MPrS]); spheres, infinite cylinders and planes are characterized as (single) invariant equipotential surfaces in  $\mathbb{R}^3$  ([MPeS]); similar symmetry results can also be proven in the sphere and the hyperbolic space ([MS4]).

In [Al2], G. Alessandrini re-considered Klamkin's problem for a bounded domain in the case in which  $u$  initially equals any function  $u_0 \in L^2(\Omega)$  and is zero on  $\partial\Omega$  for all times. He discovered that either  $u_0$  is a Dirichlet eigenfunction or  $\Omega$  is a ball. A comparable result was obtained by S. Sakaguchi [Sa] when a homogeneous Neumann condition is in force on  $\partial\Omega$ .

The aim of this paper is to show that Klamkin's property of having invariant equipotential surfaces characterizes a solution of the heat equation without assuming *any whatsoever* initial or boundary condition. This is the content of our main result.

**Theorem 1.1.** *Let  $\Omega \subseteq \mathbb{R}^N$  be a domain and let  $u$  be a solution of the heat equation:*

$$(1.1) \quad u_t = \Delta u \quad \text{in } \Omega \times (0, \infty).$$

*Assume that there exists a  $\tau > 0$  such that, for every  $t > \tau$ ,  $u(\cdot, t)$  is constant on the level surfaces of  $u(\cdot, \tau)$  and  $Du(\cdot, \tau) \neq 0$  in  $\Omega$ .*

*Then one of the following occurrences holds:*

- (i) *the function  $\varphi = u(\cdot, \tau)$  (and hence  $u$ ) is isoparametric, that is there exist two real-valued functions  $f$  and  $g$  such that  $\varphi$  is a solution of the following system of equations:*

$$|D\varphi|^2 = f(\varphi) \quad \text{and} \quad \Delta\varphi = g(\varphi) \quad \text{in } \Omega;$$

- (ii) *there exist two real numbers  $\lambda, \mu$  such that*

$$u(x, t) = e^{-\lambda t} \phi_\lambda(x) + \mu, \quad (x, t) \in \Omega \times [\tau, \infty),$$

*where*

$$\Delta\phi_\lambda + \lambda\phi_\lambda = 0 \quad \text{in } \Omega;$$

- (iii) *there exists a real number  $\gamma$  such that*

$$u(x, t) = \gamma t + w(x), \quad (x, t) \in \Omega \times [\tau, \infty),$$

*where*

$$\Delta w = \gamma \quad \text{in } \Omega.$$

As we shall see in Remark 2.1, the assumption on the gradient of the solution  $u$  is only technical. Without that assumption, another (trivial) occurrence would be that  $u$  is constant.

The proof of Theorem 1.1 relies on and completes those contained in [Al2] and [Sa]: there, option (iii) and the fact that initial and boundary conditions are unnecessary were overlooked.

*Isoparametric functions* are well-known in the literature; accordingly, their level surfaces are called *isoparametric surfaces* and can also be characterized as those surfaces whose principal curvatures are all constant. The classical results of T. Levi-Civita [Le] and B. Segre [Se] classify isoparametric functions in  $\mathbb{R}^N$  by their level surfaces: they can be either concentric spheres, co-axial spherical cylinders (that is cartesian products of an  $M$ -dimensional sphere by an  $(N - M - 1)$ -dimensional euclidean space,  $0 \leq M \leq N - 2$ ), or parallel hyperplanes (affine spaces of co-dimension 1). By using this fact, one can conclude fairly easily that, in Klamkin's setting, that is when (i) of Theorem 1.1 holds and  $u$  is constant on  $\partial\Omega$ , then the possible shapes of a domain  $\Omega$  can be one of the following: a ball, its exterior or a spherical annulus; a spherical cylinder, its exterior or a cylindrical annulus; a half-space or an infinite strip (see [Al2] for the simply connected case). Analogous results can be drawn in the case we impose on  $u$  a homogeneous Neumann condition (see [Sa]),

$$(1.2) \quad u_\nu = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty).$$

Thus, a caloric function that has invariant equipotential surfaces enjoys some *splitting property in space-time*, since it is always separable (either with respect to addition or to multiplication). We point out that this behaviour is not restricted to the case of the heat equation, since it also occurs for other *linear* evolution equations, such as the *wave equation*, the *Schrödinger equation* or any partial differential equation connected to the heat equation by some integral transform.

More interestingly, a similar behaviour holds for the following class of quasi-linear evolution equations:

$$(1.3) \quad u_t = \mathcal{Q}u \quad \text{in} \quad \Omega \times (0, \infty),$$

where the operator

$$(1.4) \quad \mathcal{Q}u = \sum_{i,j=1}^N a_{ij}(Du) u_{x_i x_j}$$

is elliptic and the coefficients  $a_{ij}(\xi)$  are sufficiently smooth  $\alpha$ -homogeneous functions of  $\xi \in \mathbb{R}^N$ ,  $\alpha > -1$ , that is such that

$$(1.5) \quad a_{ij}(\sigma \xi) = \sigma^\alpha a_{ij}(\xi) \quad \text{for every} \quad \xi \in \mathbb{R}^N, \quad \sigma > 0 \quad \text{and} \quad i, j = 1, \dots, N;$$

this will be shown in Theorem 2.2.

Important instances of (1.3) are the *evolution  $p$ -Laplace* equation,

$$(1.6) \quad u_t = \operatorname{div}\{|Du|^{p-2} Du\} \quad \text{in} \quad \Omega \times (0, \infty);$$

the *normalized evolution  $p$ -Laplace* equation,

$$(1.7) \quad u_t = |Du|^{2-p} \operatorname{div}\{|Du|^{p-2} Du\} \quad \text{in} \quad \Omega \times (0, \infty);$$

the (anisotropic) *evolution  $h$ -Laplace* equation,

$$(1.8) \quad u_t = \Delta_h u \quad \text{in} \quad \Omega \times (0, \infty);$$

here,

$$\Delta_h u = \operatorname{div}\{h(Du) Dh(Du)\}$$

is the so-called *anisotropic  $h$ -laplacian*, where  $h : \mathbb{R}^N \rightarrow \mathbb{R}$  is a non-negative convex function of class  $C^2(\mathbb{R}^N \setminus \{0\})$ , which is even and positively homogeneous of degree 1.

In Section 3, besides a brief discussion on the possible symmetries of the domain  $\Omega$  when initial and boundary conditions are added, we will also prove a classification theorem, related to equation (1.8), for isoparametric surfaces in the spirit of Levi-Civita and Segre's result (Theorem 3.1).

## 2. THE SPLITTING PROPERTY

The proof of Theorem 1.1 and Theorem 2.2 below are similar; however, we prefer to give them separately to draw the reader's attention on the different behaviours that depend on the parameter  $\alpha$  in (1.5).

*Proof of Theorem 1.1.* As originally observed in [Al1], the assumption on the level surfaces of  $u$  implies that, if we set  $\varphi(x) = u(x, \tau)$ , then there exist a number  $T > \tau$  and a function  $\eta : \mathbb{R} \times [\tau, T) \rightarrow \mathbb{R}$ , with

$$(2.1) \quad \eta(s, \tau) = s, \quad s \in \mathbb{R},$$

such that

$$(2.2) \quad u(x, t) = \eta(\varphi(x), t)$$

for every  $(x, t) \in \overline{\Omega} \times [\tau, T)$ ; thus, (1.1) gives that

$$\eta_{ss}(\varphi, t) |D\varphi|^2 + \eta_s(\varphi, t) \Delta\varphi = \eta_t(\varphi, t) \quad \text{in } \Omega \times [\tau, T).$$

By differentiating in  $t$  this identity, we obtain that  $\varphi$  must satisfy in  $\Omega$  the following system of equations:

$$(2.3) \quad \begin{aligned} \eta_s(\varphi, t) \Delta\varphi + \eta_{ss}(\varphi, t) |D\varphi|^2 &= \eta_t(\varphi, t), \\ \eta_{st}(\varphi, t) \Delta\varphi + \eta_{sst}(\varphi, t) |D\varphi|^2 &= \eta_{tt}(\varphi, t), \end{aligned}$$

for  $t \in [\tau, T)$ .

Notice that the necessary smoothness of the function  $\eta$  can be proved by a standard finite difference argument: in fact, one can prove that, since  $D\varphi \neq 0$ , then  $\eta \in C^\infty(I \times [\tau, T))$ , where  $I = (\inf_\Omega \varphi, \sup_\Omega \varphi)$  and  $\eta_s > 0$  on  $I \times [\tau, T)$  (see [Al1, Lemma 1], [Al2, Lemma 2.1] or [Sa, Lemma 2.1] for details).

As observed in [Sa], for the system (2.3), it is enough to consider the alternative cases in which the determinant

$$D(s, t) = \eta_s \eta_{sst} - \eta_{st} \eta_{ss}$$

is zero or not zero. In fact, if  $D(s, t) \neq 0$  at some  $(s, t) \in I \times [\tau, T)$ , then  $D \neq 0$  in an open neighborhood, say  $U \times V \subset I \times [\tau, T)$ , of  $(s, t)$ ; (2.3) then implies that

$$|D\varphi|^2 = f(\varphi) \quad \text{and} \quad \Delta\varphi = g(\varphi)$$

at least in a subdomain  $\Omega'$  of  $\Omega$ , and the expressions of  $f$  and  $g$  are given by the formulas

$$f = \frac{\eta_s \eta_{tt} - \eta_{st} \eta_t}{\eta_s \eta_{sst} - \eta_{st} \eta_{ss}}, \quad g = \frac{\eta_t \eta_{sst} - \eta_{tt} \eta_{ss}}{\eta_s \eta_{sst} - \eta_{st} \eta_{ss}};$$

clearly,  $f$  and  $g$  are analytic functions. Thus  $u(\cdot, t)$  (and hence  $\varphi$ ) is isoparametric in an open subdomain of  $\Omega$ ; by using the classification result for

isoparametric functions by Levi-Civita and Segre and the analiticity of  $\varphi$ , we can then conclude that the function  $\varphi$  is isoparametric in the whole  $\Omega$ .

Otherwise, we have  $D = 0$  in  $I \times [\tau, T)$ . Thus, since

$$\frac{\partial^2}{\partial s \partial t} \log(\eta_s) = (\eta_s)^{-2} (\eta_s \eta_{sst} - \eta_{st} \eta_{ss}) = 0 \quad \text{in } I \times [\tau, T),$$

we have that  $\log \eta_s(s, t)$  splits up into the sum of a function of  $s$  plus a function of  $t$ ; (2.1) then implies that  $\eta_s(s, t)$  only depends on  $t$ , and hence it is easy to conclude that

$$\eta(s, t) = a(t) s + b(t), \quad (s, t) \in I \times [\tau, T),$$

for some smooth functions  $a$  and  $b$  such that

$$a(\tau) = 1 \quad \text{and} \quad b(\tau) = 0.$$

Now, we now know that

$$u(x, t) = a(t) \varphi(x) + b(t)$$

is a solution of (1.1), thus (2.3) can be written as a linear system of equations:

$$(2.4) \quad \begin{aligned} a'(t) \varphi(x) - a(t) \Delta \varphi &= -b'(t), \\ a''(t) \varphi(x) - a'(t) \Delta \varphi &= -b''(t). \end{aligned}$$

The determinant of this system must be zero, otherwise  $\varphi$  would be constant (in fact, it would be that  $\varphi(x)$  is a function of  $t$ ); thus,

$$a(t) a''(t) - a'(t)^2 = 0 \quad \text{for } t \in [\tau, \infty), \quad a(\tau) = 1.$$

All solutions of this problem can be written as  $a(t) = e^{-\lambda(t-\tau)}$  for  $\lambda \in \mathbb{R}$  and, by going back to (2.4), we obtain that

$$\Delta \varphi(x) + \lambda \varphi(x) = b'(t) e^{\lambda(t-\tau)} = \gamma,$$

for some constant  $\gamma$ . Since  $b(\tau) = 0$ , we have:

$$b(t) = \gamma \frac{1 - e^{-\lambda(t-\tau)}}{\lambda} \quad \text{if } \lambda \neq 0 \quad \text{and} \quad b(t) = \gamma(t - \tau) \quad \text{if } \lambda = 0.$$

Therefore, we have obtained:

$$\begin{aligned} u(x, t) &= e^{-\lambda(t-\tau)} [\varphi(x) - \gamma/\lambda] + \gamma/\lambda \quad \text{if } \lambda \neq 0, \\ u(x, t) &= \varphi(x) + \gamma(t - \tau) \quad \text{if } \lambda = 0. \end{aligned}$$

In conclusion, by setting  $\phi_\lambda = e^{\lambda\tau}[\varphi - \gamma/\lambda]$  and  $\mu = \gamma/\lambda$  for  $\lambda \neq 0$ , we get case (ii), while setting  $w = \varphi - \gamma\tau$  for  $\lambda = 0$  yields case (iii).  $\square$

**Remark 2.1.** For the sake of simplicity, in Theorem 1.1 we assumed that  $Du \neq 0$  in the whole  $\Omega$ , in order to be able to deduce the necessary regularity for  $\eta$ . Here, we will show how that assumption can be removed.

The only possible obstruction to the regularity of  $\eta$  is the presence of level sets of  $\varphi$  whose points are all critical. Indeed  $\eta$  is always smooth in the  $t$  variable and is smooth in the  $s$  variable, for every semi-regular value  $s$  of  $\varphi$  (*i.e.* a value that is the image of at least one regular point). For every open subset of  $\Omega$  of regular points for  $\varphi$  corresponding to a semiregular value of  $\varphi$ , the above theorem still holds true even if we remove the assumption on the gradient of the solution  $u$ .

We now show that, in any case, the classification holds true globally in  $\Omega$ . For simplicity we assume that there exists only one critical level set.

Let then  $s$  be a value in the range of  $\varphi$  such that  $D\varphi(x) = 0$ , for every  $x \in \varphi^{-1}(s)$  and set  $\Omega^+ = \{x \in \Omega : \varphi(x) > s\}$ , and  $\Omega^- = \{x \in \Omega : \varphi(x) < s\}$ . Clearly the above classification holds true separately in  $\Omega^+$  and in  $\Omega^-$ . Being cases (i), (ii) and (iii) closed relations involving continuous functions, if the same case occurs both in  $\Omega^+$  and  $\Omega^-$ , then it holds in the whole domain  $\Omega$ . Thus, we are left with all the cases in which in  $\Omega^+$  and  $\Omega^-$  the solution assumes two different representations of those given in Theorem 1.1.

We proceed by direct inspection. If case (i) occurs in  $\Omega^+$  or in  $\Omega^-$ , as it has been already remarked, then  $\varphi$  extends to an isoparametric function in an open domain containing  $\Omega$ .

We then have only to study the case in which instances (ii) and (iii) are in force in  $\Omega^-$  and  $\Omega^+$ , respectively; by contradiction, we shall see that it is not possible to have a critical level set. In fact, up to sum a constant, we can assume without loss of generality that  $s \neq 0$ . As shown in [Al2, Lemma 2.2], the presence of a critical level set implies that  $\Delta\varphi(x) = 0$ , for every  $x \in \overline{\Omega^+} \cap \overline{\Omega^-}$ . Then, according to the previous computations, we get that  $0 = \Delta\varphi(x) = \lambda\varphi(x)$ , and, being  $\lambda \neq 0$ , we have that  $\varphi(x) = 0$  and thence  $s = 0$ , that is a contradiction.

We now consider the case of the quasi-linear operator (1.3). Notice that, when  $\alpha = 0$ , this also concerns (1.7) besides the heat equation. In what follows, it is useful to define the generalized gradient operator:

$$(2.5) \quad \mathcal{G}u = \sum_{i,j=1}^N a_{ij}(Du) u_{x_i} u_{x_j}.$$

**Theorem 2.2.** *Let  $\Omega \subseteq \mathbb{R}^N$  be a domain and let  $u \in C^1((0, \infty); C^2(\Omega))$  be a solution of equation (1.3),*

$$u_t = \mathcal{Q}u \quad \text{in } \Omega \times (0, \infty),$$

*where the operator  $\mathcal{Q}$ , given in (1.4), is elliptic with coefficients  $a_{ij}(\xi)$  that satisfy (1.5).*

*Assume that there exists a  $\tau > 0$  such that, for every  $t > \tau$ ,  $u(\cdot, t)$  is constant on the level surfaces of  $u(\cdot, \tau)$  and  $Du(\cdot, \tau) \neq 0$  in  $\Omega$ .*

*Then there exists a countable set of values  $a_1 < a_2 < \dots < a_i < \dots$ , such that*

$$\Omega = \bigcup_{i \in \mathbb{N}} \varphi^{-1}([a_i, a_{i+1}])$$

*and, for every subdomain  $\Omega' \subset \varphi^{-1}([a_i, a_{i+1}])$  one of the following cases occurs:*

- (i) there exist two real-valued functions  $f$  and  $g$  such that  $\varphi = u(\cdot, \tau)$  is a solution of the following system of equations:*

$$\mathcal{G}\varphi = f(\varphi) \quad \text{and} \quad \mathcal{Q}\varphi = g(\varphi) \quad \text{in } \Omega';$$

- (ii) there exist two real numbers  $\lambda, \mu$  such that*

$$u(x, t) = [1 + \lambda(t - \tau)]^{-1/\alpha} \phi_\lambda(x) + \mu, \quad (x, t) \in \Omega' \times [\tau, \infty),$$

with

$$\mathcal{Q}\phi_\lambda + \frac{\lambda}{\alpha}\phi_\lambda = 0 \quad \text{in } \Omega',$$

if  $\alpha \neq 1$ ;

$$u(x, t) = e^{-\lambda(t-\tau)}\phi_\lambda(x) + \mu, \quad (x, t) \in \Omega' \times [\tau, \infty),$$

with

$$\mathcal{Q}\phi_\lambda + \lambda\phi_\lambda = 0 \quad \text{in } \Omega',$$

if  $\alpha = 1$ ;

(iii) there exists a real number  $\gamma$  such that

$$u(x, t) = \gamma(t - \tau) + w(x), \quad (x, t) \in \Omega' \times [\tau, \infty),$$

where

$$\mathcal{Q}w = \gamma \quad \text{in } \Omega'.$$

*Proof.* The proof runs similarly to that of Theorem 1.1. We still begin by setting  $\varphi(x) = u(x, \tau)$  and  $u(x, t) = \eta(\varphi(x), t)$ , where  $\eta$  satisfies (2.1). Since  $Du \neq 0$  equation (1.3) is uniformly parabolic, by standard parabolic regularity (see, for instance [LU]), we have the necessary regularity to give sense to the following computations.

By arguing as in the proof of Theorem 1.1, we obtain the system of equations:

$$(2.6) \quad \begin{aligned} \xi(\varphi, \tau) \mathcal{Q}\varphi + \xi_s(\varphi, \tau) \mathcal{G}\varphi &= \eta_t(\varphi, \tau), \\ \xi_t(\varphi, \tau) \mathcal{Q}\varphi + \xi_{st}(\varphi, \tau) \mathcal{G}\varphi &= \eta_{tt}(\varphi, \tau), \end{aligned}$$

where  $\xi = (\eta_s)^{\alpha+1}$ .

At this point, the proof is slightly different from that of Theorem 1.1. If there exists  $(s, t)$  such that  $D(s, t) = \xi \xi_{st} - \xi_s \xi_t \neq 0$ , then  $D(u(x, t), t) \neq 0$  for  $x \in \Omega' = \{x \in \Omega : u(x, t) \in (s - \delta, s + \delta)\}$  for some  $\delta > 0$ . By setting

$$\bar{f} = \frac{\xi_s \eta_{tt} - \xi_t \eta_t}{\xi \xi_{st} - \xi_s \xi_t}, \quad \bar{g} = \frac{\xi_{st} \eta_t - \xi_s \eta_{tt}}{\xi \xi_{st} - \xi_s \xi_t},$$

case (i) holds true for the function  $u(\cdot, t)$ . The same fact holds true for  $\varphi$ , since  $u(x, t) = \eta(\varphi(x), t)$  and hence  $\varphi$  and  $u(x, t)$  share the same level sets; thus, there exist functions  $f$  and  $g$  such that  $\mathcal{G}\varphi = f(\varphi)$  and  $\mathcal{Q}\varphi = g(\varphi)$  in  $\Omega'$ .

Now, we define  $J$  as the closure of the complement in the image of  $u$  of the closure of the set  $\{s : D(s, t) \neq 0, \text{ for some } t\}$ ;  $J$  is a countable union of disjoint intervals. Let  $[a, b] \subseteq J$ ; in  $\Omega' = \varphi^{-1}([a, b])$ , we get

$$\begin{aligned} \frac{\partial^2}{\partial s \partial t} \log(\xi) &= (\xi)^{-2} (\xi \xi_{st} - \xi_t \xi_s) = 0 \quad \text{in } [a, b] \times [\tau, T), \\ \xi(s, \tau) &= 1 \quad \text{for } s \in [a, b]. \end{aligned}$$

As before, we obtain

$$\eta(s, t) = a(t)s + b(t),$$

with  $a(\tau) = 1$  and  $b(\tau) = 0$ . Proceeding as before with  $u(x, t) = a(t)\varphi(x) + b(t)$  gives the system:

$$(2.7) \quad \begin{aligned} a'(t)\varphi(x) - a(t)^{\alpha+1} \mathcal{Q}\varphi &= -b'(t), \\ a''(t)\varphi(x) - (\alpha + 1)a(t)^\alpha a'(t) \mathcal{Q}\varphi &= -b''(t). \end{aligned}$$

The determinant of this system must be zero, otherwise  $\varphi$  would be constant; thus,  $a$  must satisfy the problem

$$a^{\alpha+1}a'' - (\alpha + 1)a^\alpha(a')^2 = 0 \quad \text{in } [\tau, T), \quad a(\tau) = 1.$$

The solutions of this problem are for  $t \in [\tau, T)$

$$\begin{aligned} a(t) &= [1 + \lambda(t - \tau)]^{-1/\alpha} \quad \text{if } \alpha \neq 0, \\ a(t) &= e^{-\lambda(t-\tau)} \quad \text{if } \alpha = 0, \end{aligned}$$

for some  $\lambda \in \mathbb{R}$ , and going back to the first equation in (2.7) gives

$$\begin{aligned} \mathcal{Q}\varphi + \frac{\lambda}{\alpha}\varphi(x) &= b'(t)a(t)^{-(\alpha+1)} = \gamma \quad \text{if } \alpha \neq 0, \\ \mathcal{Q}\varphi + \lambda\varphi(x) &= b'(t)a(t)^{-1} = \gamma \quad \text{if } \alpha = 0. \end{aligned}$$

Thus, we have that

$$\begin{aligned} b(t) &= \frac{\gamma\alpha}{\lambda} \{1 - [1 + \lambda(t - \tau)]^{-1/\alpha}\} \quad \text{if } \alpha \neq 0, \\ b(t) &= \gamma \frac{1 - e^{-\lambda(t-\tau)}}{\lambda} \quad \text{if } \alpha = 0, \end{aligned}$$

for  $\lambda \neq 0$ ,

$$b(t) = \gamma(t - \tau),$$

for  $\lambda = 0$  and any  $\alpha \geq 0$ .

Therefore, (ii) follows when  $\lambda \neq 0$ , by setting  $\phi_\lambda = \varphi - \gamma\alpha/\lambda$  and  $\mu = \gamma\alpha/\lambda$  for  $\alpha \neq 0$  and  $\phi_\lambda = \varphi - \gamma/\lambda$  and  $\mu = \gamma/\alpha$  for  $\alpha = 0$ ; (iii) follows when  $\lambda = 0$ , by choosing  $w = \varphi$ .  $\square$

### 3. SYMMETRY IN SPACE

Theorems 1.1 and 2.2 thus give us a catalog of functions having time-invariant equi-potential surfaces. In particular, they inform us that the shape of their level surfaces can be quite arbitrary, as those of eigenfunctions are.

This arbitrariness significantly reduces if we add special initial and boundary conditions. For instance, if we consider the situation in Klamikin's conjecture, that is if  $\Omega$  is bounded and simply connected and we require that the solution  $u$  of (1.1) is initially 0 on  $\Omega$  and equals a non-zero constant on  $\partial\Omega$  at all times<sup>1</sup>, then the only possibility is that the level surfaces of  $u$  be concentric spheres, as proved in [Al1] and [Sa]. If we examine the case of a *non-simply connected* bounded domain, then the connected components of its boundary are spheres and must be concentric, because they are parallel. Thus, a spherical annulus is the only possibility. In a similar way we argue when  $\Omega$  is unbounded, thus obtaining the remaining possible solutions: a (solid) spherical cylinder; a hyperspace; an infinite strip; a cylindrical annulus; the exterior of a ball and of a spherical cylinder.

Actually, the presence of those initial and boundary conditions allows one to relax the stringent requirement that *all* equi-potential surfaces be time-invariant. In fact, to obtain the spherical symmetry of them it is enough to require that *only one* of them be time-invariant, provided it is the boundary of a  $C^1$ -smooth domain compactly contained in  $\Omega$  (see [MS1]). We also

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<sup>1</sup>It can be seen that that constant can be replaced by any bounded positive function that bounded away from zero on  $\partial\Omega$ .



point out that this result can be obtained under quite general regularity assumptions on  $\partial\Omega$ , as shown in [MS7]. The case in which  $\partial\Omega$  is not bounded is not settled: we mention [MS3] and [MS5] for some partial results in this direction and [MPeS] for a characterization in 3-space.

The possible non-regularity of  $\partial\Omega$ , instead, opens up the way to non-spherical solutions if a homogeneous Neumann boundary condition is assumed on  $\partial\Omega$  for all times. As shown in [Sa], in fact, truncated spherical cylinders are possible solutions, if  $\Omega$  is a Lipschitz domain. At this date, no results are known if a homogeneous Robin boundary condition is assumed:

$$u_\nu - \sigma u = 0 \quad \text{on } \partial\Omega \times (0, \infty).$$

We notice that the situations considered in Klamkin's conjecture and in [Sa] rule out the occurrence of items (ii) and (iii) in Theorem 1.1. Thus one has to deal with (i), which means that the relevant time-invariant surfaces can only be (portions) of spheres, spherical cylinders or hyperplanes. Of course, the same situation reproduces for any choice of initial and/or boundary conditions which rule out (ii) and (iii). For instance, any time-independent boundary condition coupled with a non-harmonic initial condition rule out (iii).

Circumstances analogous to those of Theorem 1.1 emerge in Theorem 2.2, where the Laplace operator is replaced by the more general quasi-linear operator  $\mathcal{Q}$  defined in (1.4) with coefficients satisfying (1.5). In particular, when case (i) of Theorem 2.2 occurs, in some instances we have to deal with the description of *anisotropic* isoparametric functions and surfaces. In the rest of this paragraph, we shall focus on the important case of the evolution  $h$ -Laplace equation (1.8); here, it is convenient to look at the function  $h$  as the *support function* of a convex body  $K \subset \mathbb{R}^N$  belonging to the class  $C_+^2$  of bodies having a  $C^2$ -smooth boundary with (strictly) positive principal curvatures. By setting

$$(3.1) \quad H = \frac{1}{2} h^2,$$

the system of equations that appears in item (i) of Theorem 2.2 can be conveniently re-written as

$$(3.2) \quad DH(D\varphi) \cdot D\varphi = f(\varphi) \quad \text{in } \Omega,$$

$$(3.3) \quad \Delta_h \varphi = g(\varphi) \quad \text{in } \Omega.$$

We also notice that, since  $H$  is 2-homogeneous, by Euler's identity (3.2) can be re-written as

$$(3.4) \quad 2H(D\varphi) = f(\varphi) \quad \text{in } \Omega.$$

We shall say that a function  $\varphi$  is *K-isoparametric* if it is a solution of (3.2)-(3.3); accordingly, its level surfaces will be called *K-isoparametric surfaces*.

**Theorem 3.1** (*K-isoparametric functions*). *Let  $K \subset \mathbb{R}^N$  be a convex body of class  $C_+^2$  and let  $h$  denote its support function.*

*Let  $\varphi \in C^2(\Omega)$  be a K-isoparametric function. Then its level surfaces are of the form*

$$(3.5) \quad Dh(\mathbb{S}^M) \times \mathbb{R}^{N-1-M}, \quad M = 0, \dots, N-1.$$

**Remark 3.2.** Equation (3.5) should be interpreted as follows:  $Dh(\mathbb{S}^M)$  is an  $M$ -dimensional submanifold of  $Dh(\mathbb{S}^{N-1}) = \partial K$  and the vector-valued function  $\psi : Dh(\mathbb{S}^M) \times \mathbb{R}^{N-1-M} \rightarrow \mathbb{R}^N$  defined by  $\psi(Dh(\nu), y) = Dh(\nu) + j(y)$ , where  $j$  is the natural inclusion of  $\mathbb{R}^{N-1-M}$  in  $\mathbb{R}^N$ , defines an embedding and its image coincides, up to homoteties, with a level surface of  $\varphi$ .

The proof of the Theorem 3.1 relies on the results obtained in [GM] and [HLMG], which generalize the classical ones of Levi-Civita and Segre ([Le], [Se]). In this new setting, the metric defined on the submanifolds (the level sets of the function  $\varphi$ ) is given by an anisotropic (non constant) operator. In [GM], the authors prove a classification theorem for hypersurfaces with constant anisotropic principal curvatures; the proof is mainly based on a Cartan-type identity which forces the  $K$ -isoparametric surface to admit at most two different values for principal curvatures.

*Proof.* Let  $\varphi(x)$  be a regular value of the function  $\varphi$ ,  $\Sigma$  the level surface  $\{y \in \Omega : \varphi(y) = \varphi(x)\}$  and let  $T_x(\Sigma)$  be the tangent space to  $\Sigma$  at  $x$ . We also introduce the  $K$ -anisotropic Weingarten operator,

$$(3.6) \quad W = D^2h \left( \frac{D\varphi}{|D\varphi|} \right) \frac{D^2\varphi}{|D\varphi|},$$

and the  $K$ -anisotropic mean curvature,

$$(3.7) \quad M = \frac{1}{h(D\varphi)} \left\{ \Delta_h \varphi - \frac{D\varphi \cdot [D^2H(D\varphi)] [D^2\varphi] D\varphi}{|D\varphi|^2} \right\}.$$

We differentiate (3.2) and (3.4) and obtain the identities (by the square brackets, we denote matrices):

$$(3.8) \quad \begin{aligned} [D^2H(D\varphi)] [D^2\varphi] D\varphi + [D^2\varphi] DH(D\varphi) &= f'(\varphi) D\varphi, \\ 2 [D^2\varphi] DH(D\varphi) &= f'(\varphi) D\varphi, \\ [D^2H(D\varphi)] [D^2\varphi] D\varphi &= [D^2\varphi] DH(D\varphi). \end{aligned}$$

After straightforward computations, from the definition (3.7), the identities (3.3), (3.4) and (3.8) imply that

$$M = \frac{g(\varphi) - f'(\varphi)/2}{\sqrt{f(\varphi)}},$$

that means that  $M$  is constant on  $\Sigma$ .

We are now going to show that  $M$  is actually the trace of the  $K$ -anisotropic Weingarten operator; to do this we prove the following identity

$$(3.9) \quad W(x) = \frac{D^2H(D\varphi(x)) D^2\varphi(x)}{h(D\varphi(x))} \quad \text{on } T_x(\Sigma);$$

in other words, we show that the two matrices coincide as bilinear forms on  $T_x(\Sigma)$ .

In fact, (3.1) and the homogeneities of  $h$ ,  $DH$  and  $D^2H$  imply that

$$\begin{aligned} \frac{[D^2h(\nu)][D^2\varphi]}{|D\varphi|} &= \frac{[D^2H(\nu)][D^2\varphi]}{h(\nu)|\nabla\varphi|} - \frac{[DH(\nu) \otimes DH(\nu)][D^2\varphi]}{h(\nu)^3|\nabla\varphi|} = \\ &= \frac{[D^2H(D\varphi)][D^2\varphi]}{h(D\varphi)} - \frac{[DH(D\varphi) \otimes DH(D\varphi)][D^2\varphi]}{h(D\varphi)^3} = \\ &= \frac{[D^2H(D\varphi)][D^2\varphi]}{h(D\varphi)} - \frac{1}{2}g'(\varphi) \frac{[DH(D\varphi) \otimes D\varphi]}{h(D\varphi)^3}, \end{aligned}$$

where, in the last equality, we used the second identity in (3.8). The desired formula (3.9) is then obtained by noticing that  $T_x(\Sigma)$  lies in the kernel of  $[DH(D\varphi(x)) \otimes D\varphi(x)]$ , being orthogonal to  $D\varphi(x)$ .

Notice that  $M$  only depends on the geometry of the level surface; indeed,  $\nu(x) = D\varphi(x)/|D\varphi(x)|$  is the normal unit vector to  $\Sigma$  at  $x$  and the restriction of  $-[D^2\varphi(x)]/|D\varphi(x)|$  to  $T_x(\Sigma)$  is the shape operator of  $\Sigma$ .

Now, we claim that there exist a relatively compact neighborhood  $U_x \subset \Sigma$  of  $x$  and a number  $\delta > 0$  such that, for any  $y \in U_x$ , it holds that

$$(3.10) \quad \varphi(y + \tau DH(D\varphi(y))) = \varphi(x + \tau DH(D\varphi(x))),$$

for every  $0 < \tau < \delta$ .

Without loss of generality we can assume that  $f(\varphi) = 1$ ; indeed, since  $\Sigma$  is a regular level surface for  $\varphi$ , then  $f(\varphi) > 0$ ; by taking  $\psi = F(\varphi)$  with  $F$  such that  $(F')^2 = f$ , it is easy to show that  $\psi$  is another isoparametric function, with the same level surfaces of  $\varphi$ , and such that  $H(D\psi) = 1$ .

To prove our claim, we first have to show that the integral curves of  $DH(D\varphi)$  are geodesics. Let  $U_x \subset \Sigma$  be a relatively compact neighborhood of  $x$  (of course,  $D\varphi$  does not vanish on  $U_x$ ). For every  $y \in U_x$ , let  $\gamma_y(\tau)$  be the solution of the Cauchy problem

$$\gamma'_y(\tau) = DH(D\varphi(\gamma_y(\tau))), \quad \gamma_y(0) = y,$$

and let  $\delta$  be such that, for every  $y \in U_x$ ,  $\gamma_y$  remains regular on  $[0, \delta]$ . Then we have that

$$\begin{aligned} \varphi(\gamma_y(\tau)) - \varphi(y) &= \int_0^\tau D\varphi(\gamma_y(\sigma)) \cdot \gamma'_y(\sigma) d\sigma = \\ &= \int_0^\tau D\varphi(\gamma_y(\sigma)) \cdot DH(D\varphi(\gamma_y(\sigma))) d\sigma = 2 \int_0^\tau H(\varphi(\gamma_y(\sigma))) d\sigma = 2\tau, \end{aligned}$$

where we used Euler identity for  $H$  and the fact that we are assuming that  $f = 1$ .

Moreover, we compute that

$$\begin{aligned} \gamma''_y(\tau) &= [D^2H(D\varphi(\gamma_y(\tau)))] [D^2\varphi(\gamma_y(\tau))] \gamma'_y(\tau) = \\ &= [D^2H(D\varphi(\gamma_y(\tau)))] [D^2\varphi(\gamma_y(\tau))] DH(D\varphi(\gamma_y(\tau))) = \\ &= \frac{1}{2} f'(\varphi(\gamma_y(\tau))) [D^2H(D\varphi(\gamma_y(\tau)))] D\varphi(\gamma_y(\tau)) = 0, \end{aligned}$$

where we used (3.8) and the fact that we are assuming that  $f = 1$ .

Thus,

$$\gamma_y(\tau) = y + \tau DH(D\varphi(y)) \quad \text{for } 0 \leq \tau < \delta,$$

and hence

$$\varphi(y + \tau DH(D\varphi(y))) - \varphi(y) = 2\delta \quad \text{for } 0 \leq \tau < \delta,$$

which means that (3.10) holds.

Therefore, we have proved that, for every  $\tau \in [0, \delta)$ , the surfaces

$$\Sigma_\tau = \{y : \varphi(y) = \varphi(\gamma_x(\tau))\}$$

are parallel with respect to the anisotropic metric induced by  $K$ ; also, every  $\Sigma_\tau$  has constant  $K$ -anisotropic mean curvature.

Thus, (3.5) follows the results [GM, Theorem 2.1] and [GM, Theorem 1.1] recalled in the remark.  $\square$

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